I-4. MODES IN RADIAL WAVEBEAM RESONATORS

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The electromagnetic fields in beam waveguides and Fabry-Perot resonators can be described in terms of axially propagating reiterative beam modes having a cross-sectional field distribution which can be reconstituted at periodic intervals. In the resonator case, the period of iteration is one round trip of the beam between the two reflectors. The iteration is accomplished either by diffraction effected by limiting the beam cross-section or by transformation of the cross-sectional phase distribution of the beam. The first case applies to the iris-type (Reference 1) beam waveguides and to Fabry-Perot (Reference 2) resonators with plane reflectors and the second to lens-type beam waveguides (References 3 and 4) and to resonators with spherical reflectors (References 2 and 5).

This paper is concerned with ring-shaped resonators as shown in Figures 1 and 2 whose field can be described by beam modes with similar properties but with radial rather than axial propagation. Postulating such modes the resonant field can be physically explained as follows:

Consider a beam which originates at the inner surface of the circular reflector strip and converges toward the axis of the resonator. After crossing the center area the beam diverges and returns to the reflector. The condition for resonance is that the field of the returning beam when reflected on the circular strip has the same cross-sectional amplitude and phase distribution as the original beam. Then the reflected beam can be identified with the original beam and the assumed state of excitation

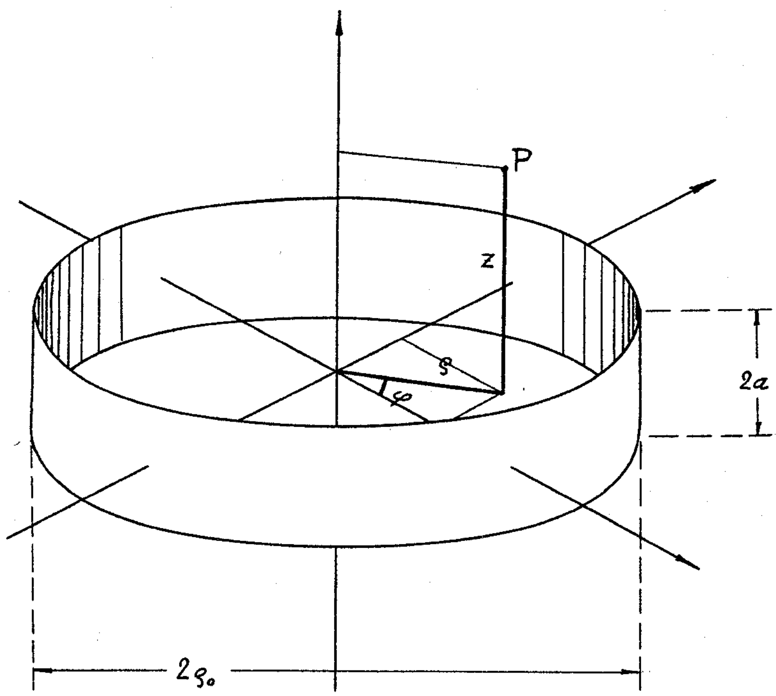

Figure 1. Ring-Shaped Resonator

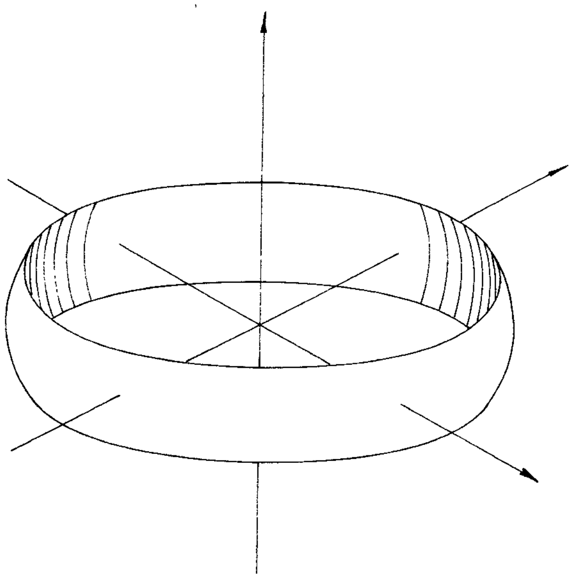

Figure 2. Ring-Shaped Resonator Curved within the Constant φ Plane

is sustained, disregarding the diffraction loss due to the small fraction of energy by-passing the reflector. The mathematical formulation of this iteration problem leads to the following integral equation for the electric or the magnetic vector potential Φ_z (z is the unit vector of the Z-direction in the coordinate system of Figure 1)

$$\Phi_m(\rho_o, Z) = p \int_{-\alpha}^{+\alpha} \Phi_m(\rho_o, \zeta)\, K(\rho_o, Z, \zeta)\, d\zeta$$

$$K(\rho_o, Z, \zeta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{H_m^{(2)}(\gamma\rho_o)}{H_m^{(1)}(\gamma\rho_o)}\, e^{-ih(Z-\zeta)}\, dh\, ,$$

where

$$\gamma^2 + h^2 = k^2 .$$

The Hankel functions of the first and second kind are $H_m^{(1)}$, $H_m^{(2)}$ respectively. The diameter of the ring reflector is $2\rho_o$ and 2a is its width. The symbol $|p|$ is the amplitude ratio between the incident and returning beam, $p \cdot p^*$ determines the diffraction loss of the reflector. If the beam is considered as a bundle of elementary plane waves, h and γ are determined by the angles of the direction of propagation against the Z - axis

$$\gamma = k \sin \alpha, \qquad h = k \cos \alpha$$

Small reflection losses require the beam to contain essentially only plane waves with propagation angles α close to 90 degrees. If one furthermore assumes that the radius ρ_o is very large compared to the wave length, the kernel of the integral equation reduces to

$$K(\rho_o, Z, \zeta) = \frac{1}{2}\sqrt{\frac{k}{\pi\rho_o}}\, e^{-i\left\{2k\rho_o - (m+3/4)\pi\right\}}\, e^{-i \frac{k}{4\rho_o} (Z-\zeta)^2}$$

With this kernel the integral equation is identical with that of the infinite parallel strip reflector treated by Fox and Li (Reference 2).

Of more practical interest are resonators which are curved within the constant φ planes as shown in Figure 2. If the radius of curvature is R, the kernel is modified to

$$K(\rho_o, Z, \zeta) = \frac{1}{2}\sqrt{\frac{k}{\pi\rho_o}}\, e^{-i\left\{2k\rho_o - (m+3/4)\pi\right\}}\, e^{-i\left\{\frac{k}{4\rho_o}(Z-\zeta)^2 - \frac{k}{2R}(Z^2+\zeta^2)\right\}}$$

In the special case $R = 2\rho_o$, the kernel reduces to a Fourier kernel whose eigen-functions are prolate spheroidal wave functions. This case is of particular interest in that the diffraction loss is at a minimum for given resonator dimensions ρ_o and a. If $k \cdot \frac{\alpha^2}{\rho_o}$ is sufficiently large the field distribution inside the resonator can be described for the various modes by the potential function

$$\Phi_{m,n}(\rho, Z) = (1+u^2)^{-\frac{1}{4}}\, He_n\left(\frac{v}{\sqrt{1+u^2}}\right) e^{-\frac{1}{4}\frac{v^2}{1+u^2}} \cdot$$

$$\cdot\, H_m^{(1)}(k\rho)\, e^{i\left[\frac{1}{4}\frac{v^2 u}{1+u^2} - (n+\frac{1}{2})\operatorname{arctg} u\right]} + H_m^{(2)}(k\rho)\, e^{-i\left[\frac{1}{4}\frac{v^2 u}{1+u^2} - (n+\frac{1}{2})\operatorname{arctg} u\right]}$$

$$\cdot\, \cos(m\phi + \beta_m)$$

With $v = \sqrt{\frac{2k}{\rho_o}}\, Z,\quad u = \frac{2}{\pi}\frac{1}{k\rho_o}\left\{H_m^{(1)}(k\rho)\, H_m^{(2)}(k\rho)\right\}^{-1},\quad \beta_m = 0, \frac{\pi}{2}$

For $\rho \to 0$

$$\Phi_{m,n}(\rho\, Z) \to \frac{1}{2^{m-1}\, m!}\, He_n(v)\, e^{-\frac{1}{4}v^2}\, (k\rho)^m \cos(m\phi + \beta_m)\,.$$

The field components are derived from the electric and from the magnetic vector potentials $\Phi \cdot z$ in the usual manner. The resonant wave lengths $\lambda_{l,m,n}$ are given by

$$\frac{2\rho_o}{\lambda_{l,m,n}} = l + \frac{m}{2} + \frac{n}{4} + \begin{cases} \frac{7}{8} & \text{for the modes derived from the electric vector potential} \\ \frac{3}{8} & \text{for the modes derived from the magnetic vector potential} \end{cases}$$

where l,m and n are integers.

The field distribution of these modes, their orthogonality relations and their radiation characteristics (if the reflector is partially transparent) will be discussed.

REFERENCES

1. Christian, J. R and Goubau, G., "Some Measurements on an Iris Beam Waveguide," Proc. IRE, Vol. 49, pp. 1679-80, November 1961.
2. Fox. A. G. and Li, T., "Resonant Modes in a Maser Interferometer," Bell Syst. Techn. Journal, Vol. 40, pp. 453-488, March 1961.
3. Christian, J. R. and Goubau, G., "Experimental Studies on a Beam Waveguide for Millimeter Waves," IRE Trans. on Antennas and Prop., Vol. AP-9, pp. 256-263, May 1961.
4. Goubau, G. and Schwering, F., "On the Guided Propagation of Electromagnetic Wave Beams," IRE Trans. on Antennas and Prop., Vol. AP-9, pp. 248-256, May 1961.
5. Boyd, G. D. and Gordon, J P., "Confocal Multimode Resonator for Millimeter through Optical Wavelength Masers," Bell Syst. Techn. Journal, Vol. 40, pp.489-508, March 1961.